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Rational Ruled Surfaces Passing Through Two Fixed Lines

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Abstract. For any positive integer n, a rational ruled surface of degree $n+1$ is constructed which passes through two arbitrary skew lines in the three-dimensional Euclidean space. In the cases of two parallel or intersecting lines, a rational ruled surface of degree $2n+1$ is constucted which contains the lines. Any surface is a preimage of a plane under a birational space transformation. This interpretation gives implicit equations and parametric representations of the considered surfaces.

§1. Introduction

Ruled surfaces play an important role in computer aided geometric design (see [2,5,6]). In this paper, we construct rational ruled surfaces which are generalizations of the hyperbolic paraboloid and the hyperboloid of one sheet. Our main results describe three families of ruled surfaces which pass through two skew lines, two parallel lines, and two intersecting lines. The resulting surfaces in three-dimensional Euclidean space $\mathbb{R}^3$ (especially their parts without singularities) can be used in engineering and manufacturing. These ruled surfaces are found by the use of some birational transformations of $\mathbb{R}^3$. This interpretation also provides a way for finding an implicit equation and parametric representation of any such surface.

The paper is organized as follows. We introduce special birational transformations of the projective space $\mathbb{P}^3$ in Section 2. Any transformation determines a three-parameter family of surfaces whose images are planes. Corresponding transformations and rational surfaces in $\mathbb{R}^3$ are described in Section 3. For any positive integer n, a rational ruled surface of degree $n+1$ is constructed which passes through two arbitrary skew lines in the next section. The rational ruled surfaces of odd degree passing through two parallel or intersecting lines are considered in the last two sections.

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§2. Birational Transformations of the Projective 3-Space

Any rational surface is birational equivalent to a plane. In particular, the preimage of a plane under a birational space transformation is a rational surface. In this section, we shall consider birational transformations such that the preimage of an arbitrary plane is a rational ruled surface. First, we briefly recall some basic notions for birational transformations.

Definition 1. *Let $\mathbb{P}^3$ be the three-dimensional complex projective space. The map of $\mathbb{P}^3$ into itself*

$$T\,:\,\mathbb{P}^3 \to \mathbb{P}^3$$

is called a birational transformation *if there exists an open subset $U \subset \mathbb{P}^3$ in the Zariski topology such that the restriction $T|_U\,:\,U \to U$ is a one-to-one correspondence.*

In terms of homogeneous coordinates, the map T is birational, if

i) T is given by the equations

$$\rho' X'_i = F'_i(X_0, X_1, X_2, X_3), \qquad i = 0, 1, 2, 3, \tag{1}$$

where F'_i are homogeneous polynomials of the same degree and ρ' is a nonzero factor of proportionality;

ii) The inverse map T^{-1} exists and is given by the equations

$$\rho'' X''_i = F''_i(X_0, X_1, X_2, X_3), \qquad i = 0, 1, 2, 3, \tag{2}$$

where $F^{"}_i$ are also homogeneous polynomials of the same degree and $\rho'' \neq 0$.

In (1) and (2), the quadruples (X_0, X_1, X_2, X_3), $(X'_0, X'_1,\ X'_2, X'_3)$ and $(X''_0, X''_1, X''_2, X''_3)$ are homogeneous coordinates of the points $q \in \mathbb{P}^3$, $q' = T(q)$ and $q'' = T^{-1}(q)$, respectively. Moreover, it is possible that $\deg F'_i \neq \deg F''_i$.

A fundamental (or base) locus of the birational transformation T given by (1) is the variety of common zeros of the polynomials F'_i. There is a three-parameter family of rational surfaces such that the image of any surface under T is a plane. Then, the intersection of all such surfaces is the fundamental locus of T. Note that the birational transformations of the projective space are also called Cremona transformations. More information for the birational transformarions can be found in [1] and [4].

Now, we shall study a class of birational space transformations. For any three fixed numbers $\theta_1 \in \mathbf{C} \setminus \{0\}$, $\theta_2 \in \mathbf{C} \setminus \{0\}$ and $\theta \in \mathbf{C} \setminus \{0, 1\}$, and for any positive integer n, we may consider the map $T_0\,:\,\mathbb{P}^3 \to \mathbb{P}^3$ given by the equations

$$\begin{aligned}
\rho X'_0 &= (\theta_1 X_0^n - \theta_2 X_3^n) X_0,\\
\rho X'_1 &= (\theta_1 X_0^n - \theta_2 X_3^n) X_1 - (1-\theta)\theta_1 (X_1 X_0^n - X_2 X_3^n),\\
\rho X'_2 &= (\theta_1 X_0^n - \theta_2 X_3^n) X_2 - (1-\theta)\theta_2 (X_1 X_0^n - X_2 X_3^n),\\
\rho X'_3 &= (\theta_1 X_0^n - \theta_2 X_3^n) X_3.
\end{aligned} \tag{3}$$

From the condition $\theta \neq 1$, it follows that T_0 is not the identity mapping.

Theorem 1. *The map T_0 is a birational transformation, and the reducible curve*

$$B : \begin{cases} \theta_1 X_0^n - \theta_2 X_3^n &= 0 \\ X_1 X_0^n - X_2 X_3^n &= 0 \end{cases}$$

is the fundamental locus of T_0.

Proof: Let C be the surface given by the equation $\theta_1 X_0^n - \theta_2 X_3^n = 0$. Then, the map T_0 is a one-to-one correspondence in the set $\mathbb{P}^3 \setminus C$ which is open in the Zariski topology. On the other hand, the inverse mapping T_0^{-1} is defined by the equations (3) in which θ is replaced by θ^{-1}. Thus, T_0 is birational. The curve B is the set in which T_0 is not defined. Hence, B is the fundamental locus of T_0. □

The linear transformations of $\mathbb{P}^n$ $(n = 1, 2, 3)$ and their invariant, a cross-ratio, are studied in detail in [7]. Some geometric properties of the nonlinear transformation T_0 can be described in terms of a cross-ratio and collineations. The line $S_{03} \subset \mathbb{P}^3$ given by $X_0 = X_3 = 0$ is the $n-$fold line of the ruled surface G given by the equation $X_1 X_0^n - X_2 X_3^n = 0$. Let c be the point with homogeneous coordinates $(0, \theta_1, \theta_2, 0)$. Then, for any point $q \in \mathbb{P}^3 \setminus C$, the joining line $\overline{cq}$ meets G at the second point $q^0 \neq c$. From here, the point $q' = T_0(q)$ lies on the line $\overline{cq}$, and the cross-ratio $\{c, q^0; q, q'\} = \theta$. Continuing in this way, we consider a plane P given by $\lambda_0 X_0 + \lambda_3 X_3 = 0$, where $(\lambda_0, \lambda_3) \in \mathbf{C}^2 \setminus \{(0,0)\}$ and $\lambda_0^n : \lambda_3^n \neq \theta_1 : \theta_2$. Then, the intersection $P \cap G$ falls into $n-$fold line S_{03} and another line H not containing the point c. From the equations (3), we may conclude that T_0 preserves P and the restriction $T_0|_P : P \to P$ is a plane homology with a vertex c, an axis H and a modulus θ. It is clear that the set of all fixed points of T_0 is $G \setminus B$.

Theorem 2. *Let V be a surface in $\mathbb{P}^3$ such that the image $T_0(V)$ is a plane. Then, V is a rational ruled surface of degree $n+1$. In the case $n \geq 2$, the singular locus of V is an $n-$fold line S_{03}.*

Proof: From the (3) it follows that the surface V is given as the locus of

$$(\theta_1 X_0^n - \theta_2 X_3^n)(\sum_{i=0}^{3} \lambda_i X_i) - (1-\theta)(\lambda_1 \theta_1 + \lambda_2 \theta_2)(X_1 X_0^n - X_2 X_3^n) = 0, \quad (4)$$

where $\lambda_i \in \mathbf{C}$ for $i = 0, 1, 2, 3$ and $\sum_{i=0}^{3} |\lambda_i| \neq 0$. Hence, $\deg V = n + 1$. By (4), if P is a plane through the line S_{03}, then the intersection $P \cap V$ falls into the $n-$fold line S_{03} and another line L. This means that V is a ruled surface. It is known from [3] that the singular locus of a ruled surface is connected. Thus, $\mathrm{Sing}(V) = S_{03}$. □

§3. Birational Transformations of the Euclidean 3-Space

Suppose that θ_1, θ_2 and θ are nonzero real numbers. Then, by (3), this defines a birational transformation of the 3-dimensional real projective space, or equivalently, a birational transformation of the projective extension of the Euclidean 3-space. Thus, we get a birational transformation of the Euclidean 3-space.

Theorem 3. *Let n be a positive integer, θ_1 and θ_2 be positive real numbers, and let $\theta \in \mathbb{R} \setminus \{0, 1\}$. Then, the transformation $T_1 : \mathbb{R}^3 \to \mathbb{R}^3$ given by the equations in Cartesian coordinates*

$$\begin{aligned} x' &= \{(\theta_1 - \theta_2 z^n)x - (1-\theta)\theta_1(x - yz^n)\}(\theta_1 - \theta_2 z^n)^{-1}, \\ y' &= \{(\theta_1 - \theta_2 z^n)y - (1-\theta)\theta_2(x - yz^n)\}(\theta_1 - \theta_2 z^n)^{-1}, \\ z' &= z, \end{aligned} \tag{5}$$

is birational. If V is a surface in $\mathbb{R}^3$ and the image $T_1(V)$ is a plane, then V is a rational ruled surface of degree $n+1$, and there is an unique generatrix of V in any plane P_t given by the equation $z - t = 0$ $(t \in \mathbb{R})$.

Proof: Substituting $X_1 X_0^{-1} = x$, $X_2 X_0^{-1} = y$ and $X_3 X_0^{-1} = z$ (into the equations (3)), we obtain (5). Then, the statements follow from Theorem 1 and Theorem 2. □

The inverse transformation T_1^{-1} is defined by (5), where θ is replaced by θ^{-1}. Hence, both T_1 and T_1^{-1} are not defined at the points of the reducible surface given by the equation $\theta_1 - \theta_2 z^n = 0$. Any surface V contains the reducible curve B_1 given by $\theta_1 - \theta_2 z^n = x - yz^n = 0$ which is the fundamental locus of T_1. If two lines are components of the curve B_1, then there is a family of rational ruled surfaces V passing through the lines. We shall use this property in the next sections.

Definition 2. *We say that the type of the surface $V \subset \mathbb{R}^3$ is $\mathcal{HP}_{n+1}$ if its transform $T_1(V)$ is a plane.*

It is clear that in the case $n = 1$, V is a hyperbolic paraboloid.

§4. Skew Lines

In this section we fix two skew lines L_1 and L_2 in the Euclidean space $\mathbb{R}^3$. Let φ be the angle between the lines L_1 and L_2, and let d be the distance between the same lines. Without loss of generality, we suppose that $0 < \varphi \le \frac{1}{2}\pi$ and $\tan\varphi \ne d^n$. The mutual position of L_1 and L_2 is completely determined by φ and d.

Theorem 4. *For any positive integer n, there exists a two-parameter family of rational ruled surfaces which meet the following requirements:*
i) *The type of any surface is $\mathcal{HP}_{n+1}$,*
ii) *Any surface passes through L_1 and L_2.*

Proof: Let (x, y, z) be Cartesian coordinates in $\mathbb{R}^3$. Then, we may assume that

$$L_1 : \begin{cases} z - d &= 0 \\ x - d^n y &= 0 \end{cases} \quad \text{and} \quad L_2 : \begin{cases} z &= 0 \\ kx - y + 1 &= 0, \end{cases}$$

where $k = \frac{d^n \sin\varphi + \cos\varphi}{d^n \cos\varphi - \sin\varphi}$. Consider the birational transformation T_1 defined by (3) in which $\theta_1 = d^n$ and $\theta_2 = 1$. If the surface $V \subset \mathbb{R}^3$ is given as locus of

$$(d^n - z^n)(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 z) - (1-\theta)(\lambda_1 d^n + \lambda_2)(x - yz^n) = 0, \quad (6)$$

where $\lambda_i \in \mathbb{R}$ $(i = 0,1,2,3)$ and $\sum_{i=0}^{3} |\lambda_i| \neq 0$, then the image $T_1(V)$ is a plane. Let $\lambda_0 = 1$, $\lambda_1 = k\,\theta^{-1} + (1-\theta^{-1})d^{-n}$, $\lambda_2 = -1$, and $\lambda_3 = \mu \in \mathbb{R}$. Thus, we obtain a 2-parameter family of rational ruled surfaces $V(\theta, \mu)$ given by

$$(d^n - z^n)\{1 + (\theta^{-1}k + \frac{1-\theta^{-1}}{d^n})x - y + \mu z\} \\ + (1-\theta^{-1})(d^n k - 1)(x - yz^n) = 0.$$

The pencil of lines on $V(\theta,\mu)$ can be represented as $L(t) = P_t \cap Q_t$ $(t \in \mathbb{R})$, where the plane P_t is given by $z - t = 0$ and the plane Q_t is given by

$$(d^n - t^n)\{1 + (\theta^{-1}k + \frac{1-\theta^{-1}}{d^n})x - y + \mu t\} \\ + (1-\theta^{-1})(d^n k - 1)(x - yt^n) = 0.$$

Then, $L_1 = L(t = d)$ and $L_2 = L(t = 0)$. □

Now, we can obtain a parametric representations of the above surfaces. If $\theta \neq 0, 1$ and μ are fixed real numbers, then the parametric equations of the surface $V(\theta, \mu)$ are

$$x = u \qquad y = \frac{f_1(u,t)}{g_1(t)} \qquad z = t,$$

where u and t are real parameters,

$$f_1(u,t) = (t^d - d^n)\{1 + (\theta^{-1}\,k + d^{-n} - \theta^{-1}\,d^{-n})u + \mu t\} \\ + (1-\theta^{-1})(1 - k\,d^n)u$$
$$g_1(t) = t^n - d^n + (1-\theta^{-1})(1 - k\,d^n)t^n.$$

§5. Parallel Lines

Using the birational transformations defined in Section 3, we can construct noncylindrical rational ruled surfaces of odd degree which pass through two fixed parallel lines.

Theorem 5. *Let L_1 and L_2 be two parallel lines in the Euclidean space $\mathbb{R}^3$. Then, for any positive integer m, there exists a four-parameter family of rational ruled surfaces which meet the following requirements:*
i) *The type of any surface is $\mathcal{HP}_{2m+1}$;*
ii) *Any surface passes through L_1 and L_2.*

Proof: Let $2d$ be the distance between L_1 and L_2. Then, we may suppose that

$$L_j : \begin{cases} z + (-1)^j d & = 0 \\ z - d^{2m} y & = 0, \end{cases} \qquad j = 1, 2.$$

Consider the transformation T_1 given by the equations (3), in which $\theta_1 = d^{2m}$, $\theta_2 = 1$ and $n = 2m$. If $V \subset \mathbb{R}^3$ is a surface such that the image $T_1(V)$ is a plane, then V is given as locus of

$$(d^{2m} - z^{2m})(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 z) - (1-\theta)(\lambda_1 d^{2m} + \lambda_2)(x - yz^{2m}) = 0,$$

where $\lambda_i \in \mathbb{R}$ and $\sum_{i=0}^{3} |\lambda_i| \neq 0$. For $t \in \mathbb{R}$, let P_t be the plane given by the equation $z - t = 0$ and Q_t be the plane given by the equation

$$(d^{2m} - t^{2m})(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 t) - (1-\theta)(\lambda_1 d^{2m} + \lambda_2)(x - yt^{2m}) = 0.$$

Then the one-parameter family of lines $L(t) = P_t \cap Q_t \subset V$ contains the lines $L_1 = L(t = d)$ and $L_2 = L(t = -d)$. □

The above description of the generatrices of the surface V also gives its parametric equations

$$\begin{aligned} x &= u, \\ y &= \frac{(t^{2m} - d^{2m})(\lambda_0 + \lambda_1 u + \lambda_3 t) + (1-\theta)(\lambda_1 d^{2m} + \lambda_2)u}{\lambda_2(d^{2m} - t^{2m}) + (1-\theta)(\lambda_1 d^{2m} + \lambda_2)\, t^{2m}}, \\ z &= t, \end{aligned}$$

where u and t are real parameters.

Finally, we observe a special property of the surface $V \in \mathcal{HP}_{2m+1}$. From the proof of Theorem 5, it follows that the lines $L(t)$ and $L(-t)$ are parallel for any $t \neq 0$. Moreover, if $t_1 \neq t_2$ and $t_1 \neq -t_2$, then $L(t_1)$ and $L(t_2)$ are skew lines. In other words, the rational ruled surface $V \in \mathcal{HP}_{2m+1}$ is noncylindrical.

§6. Intersecting Lines

First, we consider another interpretation in $\mathbb{R}^3$ of the birational transformation T_0. Next, using this interpretation, we construct a four-parameter family of nonconic rational ruled surfaces which pass through two fixed intersecting lines.

Theorem 6. *Let ψ be an acute angle, and let $T_2 : \mathbb{R}^3 \to \mathbb{R}^3$ be the transformation given by the equations*

$$\begin{aligned} x' &= \frac{x\{h(x,y) - z^n\} - (1-\theta)\{xh(x,y) - yz^n\}}{h(x,y) - z^n - \sqrt{2}(1-\theta)\{xh(x,y) - yz^n\}\tan\psi} \\ y' &= \frac{y\{h(x,y) - z^n\} - (1-\theta)\{xh(x,y) - yz^n\}}{h(x,y) - z^n - \sqrt{2}(1-\theta)\{xh(x,y) - yz^n\}\tan\psi} \\ z' &= \frac{z\{h(x,y) - z^n\}}{h(x,y) - z^n - \sqrt{2}(1-\theta)\{xh(x,y) - yz^n\}\tan\psi}, \end{aligned} \tag{7}$$

where $\theta \in \mathbb{R} \setminus \{0,1\}$ and $h(x,y) = \{1 - \frac{\tan\psi}{\sqrt{2}}(x+y)\}^n$. Then, T_2 is birational. If $W \subset \mathbb{R}^3$ is a surface such that the image $T_2(W)$ is a plane, then W

is a rational ruled surface of degree $n+1$, and the singular locus of W is the n-fold line

$$S : \begin{cases} z & = 0 \\ x + y - \frac{\sqrt{2}}{\tan\psi} & = 0 \end{cases} .$$

Proof: Substituting into (3) $\theta_1 = \theta_2 = 1$, $\frac{X_1}{X_0+X_1+X_2} = \frac{x\tan\psi}{\sqrt{2}}$, $\frac{X_2}{X_0+X_1+X_2} = \frac{y\tan\psi}{\sqrt{2}}$, $\frac{X_3}{X_0+X_1+X_2} = z$, we get (7). Thus, the statement follows from Theorem 1 and Theorem 2. □

Definition 3. *We say that the type of a surface $W \subset \mathbb{R}^3$ is $\mathcal{HOS}_{n+1}$, if the transform $T_2(W)$ is a plane.*

Note that in the case $n = 1$, W is a hyperboloid of one sheet.

Theorem 7. *Let L_1 and L_2 be two intersecting lines in $\mathbb{R}^3$, and let m is a positive integer. Then, there is a four-parameter family of rational ruled surfaces such that the type of any surface is $\mathcal{HOS}_{2m+1}$ and any surface passes through L_1 and L_2.*

Proof: Let 2ψ be the angle between L_1 and L_2. Then, we may suppose that

$$L_j : \begin{cases} x - y & = 0 \\ (x+y)\tan\psi + (-1)^{j+1}\sqrt{2}\,z - \sqrt{2} & = 0 \end{cases} \quad j = 1, 2.$$

Consider the transformation T_2 given by the equation (7) in which $n = 2m$. If $W \subset \mathbb{R}^3$ is a surface and the image $T_2(W)$ is a plane, then W is given by

$$\begin{aligned} &(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 z)\{h(x,y) - z^n\} \\ &- (1-\theta)(\sqrt{2}\,\lambda_0 \tan\psi + \lambda_1 + \lambda_2)\{xh(x,y) - yz^n\} = 0, \end{aligned} \tag{8}$$

where $\lambda_i \in \mathbb{R}$, $\sum_{i=0}^{3} |\lambda_i| \neq 0$. Let $\overline{P}_t \quad (t \in \mathbb{R})$, be a plane given by

$$1 - \frac{\tan\psi}{\sqrt{2}}(x+y) = 0 \tag{9}$$

and let $\overline{Q}_t \quad (t \in \mathbb{R})$ be a plane given by

$$\begin{aligned} &(\lambda_0 + \lambda_1 x + \lambda_2 y + \lambda_3 t)(1 - t^{2m}) \\ &- (1-\theta)(\sqrt{2}\,\lambda_0 \tan\psi + \lambda_1 + \lambda_2)\{x - y\,t^{2m}\} = 0. \end{aligned} \tag{10}$$

Then, the line $L(t) = \overline{P}_t \cap \overline{Q}_t$ lies on W for any $t \in \mathbb{R}$. It is easy to see that $L_1 = L(t=1)$ and $L_2 = L(t=-1)$. Hence, W is a ruled surface which contains L_1 and L_2. □

Equation (8) with the additional conditions $x \leq 0$ and $y \leq 0$ determines a domain on the surface W which is smooth and without singularities. Other properties of W follow from (9) and (10). The line $L(t)$ intersects the line $L(-t)$ at a point on $n-$fold line S for any $t \in \mathbb{R}\setminus\{0\}$. If $t_1 \neq t_2$ and $t_1 \neq -t_2$, then $L(t_1)$ and $L(t_2)$ are skew lines. This means that W is a nonconical surface.

Finally, the parametric representation of the ruled surface W is

$$x = u, \qquad y = \frac{f_2(u,t)}{g_2(t)}, \qquad z = t\{1 + \frac{\tan\psi}{\sqrt{2}}(u + \frac{f_2(u,t)}{g_2(t)})\},$$

where u and t are real parameters, and

$$\begin{aligned} f_2(u,t) &= (1-\theta)(\sqrt{2}\tan\psi\,\lambda_0 + \lambda_1 + \lambda_2) - (\lambda_0 + \lambda_1\,u + \lambda_3\,t)(1 - t^{2m}), \\ g_2(t) &= (1-\theta)(\sqrt{2}\tan\psi\,\lambda_0 + \lambda_1 + \lambda_2)\,t^{2m} + \lambda_2(1 - t^{2m}). \end{aligned}$$

Using the general properties of the birational transformations, we get the implicit equations of the surfaces in the last three sections. But to find the parametric representations of these surfaces, we apply the specific properties of the transformations T_1 and T_2.

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